

PHILOSOPHICAL
TRANSACTIONS.

I. *Calculations to determine at what Point in the Side of a Hill its Attraction will be the greatest, &c.*

By Charles Hutton, LL.D. and F. R. S.

*In a Letter to Nevil Maskelyne, D.D. F. R. S.
and Astronomer Royal.*

Read Nov. 11, 1779.

DEAR SIR,

Royal Military Acad.
Sept. 21, 1779.

AS the experiment of determining the universal attraction of matter, which you lately conducted with so much accuracy and success, is of so great importance

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portance and curiosity, that it may probably be hereafter repeated by the learned in other countries, and in other situations; and as the utmost precision is desirable in so nice an experiment; I have composed the inclosed paper to determine the best part of a hill for making the observations, so as to obtain the greatest quantity of attraction.

I have no doubt, Sir, that the determination of this point will appear of some consequence in the opinion of one who has the improvement of useful knowledge so much at heart; and if the manner in which it is here made, meet your approbation, I shall desire the favour of your presenting it to the Royal Society.

I have the honour to be, &c.

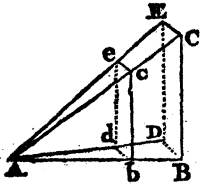
1. The

1. The great success of the experiment, lately made by the Royal Society, on the hill Schehallien, to determine the universal attraction of matter, and the important consequences that have resulted from it, may probably give occasion to other experiments of the same kind to be made elsewhere: and as all possible means of accuracy and facility are to be desired in so delicate and laborious an undertaking; it has occurred to me that it might not be unuseful to add, by way of supplement to my paper of calculations relative to the above-mentioned experiment, an investigation of the height above the bottom of a hill, at which its horizontal attraction shall be the greatest; since that is the height at which commonly the observations ought to be made, and since this best point of observation has never been any where determined that I know of, but has been variously spoken of or guessed at, it being sometimes accounted at $\frac{1}{3}$, and sometimes at $\frac{1}{2}$ of the height of the hill; whereas from these investigations it is found to be generally at about only $\frac{1}{4}$ of the altitude from the bottom.

2. Let ABCEDA be part of a cuneus of matter, its sides or faces being the two similar right-angled triangles ABC, ADE meeting in the point A, and forming the indefinitely small angle BAD. Then of any section

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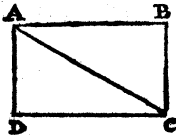
bced, perpendicular to the planes ABD and ADE, the attraction on a body at A in the direction AB, is equal to the constant quantity ss ; where $s = \sin. \angle BAC$ and $s = \sin. \angle BAD$, to the radius 1.

For, first, since the magnitude of the flowing section is every where as Ab^2 , and the attraction of the particles of matter inverfely as the fame, or as $\frac{1}{Ab^2}$; therefore their product or $\frac{Ab^2}{Ab^2}$ or 1 (a constant quantity) is as the force of attraction of bced.

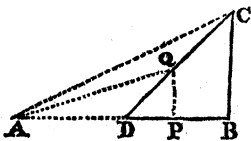
Then to find what that quantity is. Put $AB = a$, and $BC = x$; then BD or CE (the distance between the two planes at the distance AB) is $= as$. Now the force of a particle in the line CE is as $\frac{1}{AC^2}$ in the direction AC , and therefore it is as $\frac{AB}{AC^3}$ in the direction AB ; consequently the force of the whole lineola CE in the direction AB is $\frac{AB \cdot CE}{AC^3}$; and therefore the fluxion of the force of the section $BCED$ or \dot{f} is $= \frac{AB \cdot CE}{AC^3} \cdot \dot{BC} = \frac{a \cdot as \cdot \dot{x}}{a^2 + x^2} = \frac{a^2 s \dot{x}}{a^2 + x^2}$; and the fluent gives $f = \frac{sx}{\sqrt{a^2 + x^2}} = s \times \frac{BC}{AC} = ss$ for the attraction itself.

3. To

3. To find now the attraction of the whole right-angled cuneus on a body at A in the direction AB.— Since the force of each section is ss by the last article, therefore the force of all the sections, the number of them being AB or a , is $ass = s \cdot AB \cdot \frac{BC}{AC}$ the force of the whole cuneus ABCEDA.



4. To find the attraction of the rectangular part ABCD on A in the direction AB; ABCD being one side of the cuneus, and AD its edge.—Put $AD = BC = b$, and $AB = x$. Then, the force of any section as BC being always as ss by Art. 2, the fluxion of the force or f will be $= ss\dot{x} = s\dot{x} \times \frac{BC}{AC} = s\dot{x} \times \frac{b}{\sqrt{b^2 + x^2}} = \frac{bs\dot{x}}{\sqrt{b^2 + x^2}}$; and the fluent is $f = bs \times \text{hyp. log. of } \frac{x + \sqrt{b^2 + x^2}}{b} = s \cdot BC \times \text{hyp. log. } \frac{AB + AC}{BC} =$ the attraction of ABCD.

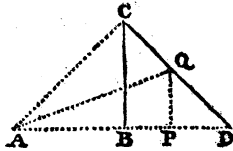


5. To find the attraction of the right-angled part BCD of a cuneus whose edge passes through A the place of the body attracted.—Put $AB = a$, $BC = b$, $BD = c$, $DA = d = a - c$, $DC = e$, $AC = g$, and $DP = x$. Then, the force of any section PQ being still as ss , the fluxion of the force of the part DPQ is $f = ss\dot{x} = s\dot{x} \times \frac{PQ}{AQ} = \frac{bsx\dot{x}}{\sqrt{c^2 d^2 + 2c^2 dx + e^2 x^2}}$; and the correct fluent

when:

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when $x = c$ is $f = \frac{bcs}{ee} \times g - d - \frac{dc}{e} \times \text{hyp. log. } \frac{ee + eg + dc}{de + dc}$
 = the force of a body at A in the direction AB.



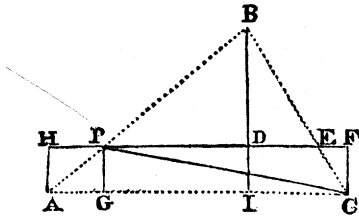
6. To find the attraction of the right-angled part BCD on the point A.—Using here again the same notation as in the last article, we have

$f = s \dot{s} \dot{x} = s \dot{x} \times \frac{PQ}{AQ} = \frac{bsx \dot{x}}{\sqrt{c^2 x^2 - 2c^2 dx + c^2 x^2}}$. The correct fluent of which, when $x = c$, is

$$f = \frac{bcs}{ee} \times g - d + \frac{dc}{e} \times \text{hyp. log. } \frac{ee + eg - dc}{de - dc}.$$

7. To apply now these premises to the finding of the place where the attraction of a hill is greatest, it will be necessary to suppose the hill to have some certain figure. That position is most convenient for observing the attraction, in which the hill is most extended in the east and west direction. Supposing then such a position of a hill, and that it is also of a uniform height and meridional section throughout; the point of observation must evidently be equally distant from the two ends. But instead of being only considerably extended, I shall suppose the hill to be indefinitely extended to the east and to the west of the point of observation, in order that the investigation may be mathematically true, and yet at the same time sufficiently exact for the before-said limited extent also. It will also come nearest to the

practical experiment, to suppose the hill to be a long triangular prism, so that all its meridional sections may be similar triangles. Let therefore the triangle ABC re-



present its section by a vertical plane passing through the meridian, or one side of an indefinitely thin cuneus whose edge is in PG; or rather PBCGP the side of one cuneus,

and PAG the side of another, their common edge being the line PG perpendicular to the base AC; P being the required point in the side AB where the attraction of the section ABC, or indefinitely thin cuneus, shall be greatest in a direction parallel to the horizon AC. And then from the foregoing suppositions, it is evident that in whatever point of AB the attraction of ABC is greatest, there also will the attraction of the whole hill be the greatest.

8. Now draw HPDEF parallel to AC; and AH, PG, BI, CF, perpendicular to the same. Then it is evident that at the point P, in the direction PF, the attraction of PBCGP is affirmative, and that of PAG negative. But PBCGP is = PBD + BDE + PFCG - EFC; and PAG = PHAG - PHA. Therefore the attractions of PBD, BDE, PFCG, RHA, are affirmative; and those of EFC, PHAG, negative.

Put

Put now $BI = a$, $AI = b$, $IC = c$, $AB = d$, $BC = e$, $AC = g = b + c$, and $PG = x$, the altitude of the point P above the bottom. Also let $s =$ the sine of the indefinitely small angle of the cuneus to rad. 1; and $q^2 = \sqrt{a^2 g^2 - 2abgx + d^2 x^2}$.

Then by Art. 3, the attraction

$$\text{of } \left\{ \begin{array}{l} \text{PBD is } s \cdot \text{PD} \cdot \frac{\text{BD}}{\text{BP}} = sb \times \frac{a-x}{d}, \\ \text{PHA is } s \cdot \text{PH} \cdot \frac{\text{PG}}{\text{PA}} = sb \times \frac{x}{d}. \end{array} \right\}$$

By Art. 4, the attraction

$$\text{of } \left\{ \begin{array}{l} \text{PFCG is } s \cdot \text{PG} \times \text{h. l. } \frac{\text{PF} + \text{PC}}{\text{PG}} = sx \times \text{h. l. } \frac{ag - bx + qq}{ax}, \\ \text{PGAH is } s \cdot \text{PG} \times \text{h. l. } \frac{\text{PH} + \text{PA}}{\text{PG}} = sx \times \text{h. l. } \frac{b+d}{a}. \end{array} \right\}$$

By Art. 5, the attraction of EFC is

$$\begin{aligned} s \cdot \frac{\text{EF} \cdot \text{FC}}{\text{EC}^2} \times \text{PC} - \text{PE} - \frac{\text{PE} \cdot \text{EF}}{\text{EC}} \times \text{h. l. } \frac{\text{EC}^2 + \text{EC} \cdot \text{PC} + \text{PE} \cdot \text{EF}}{\text{PE} \cdot \text{EC} + \text{PE} \cdot \text{EF}} \\ = \frac{sc}{ee} \times qq - g \cdot a - x - gv \cdot \frac{a-x}{e} \times \text{h. l. } \frac{acg + eqq + aax - bcx}{g \cdot c + e \cdot a - x}. \end{aligned}$$

Lastly by Art. 6, the attraction of BDE is

$$\begin{aligned} s \cdot \frac{\text{BD} \cdot \text{DE}}{\text{BE}^2} \times \text{PB} - \text{PE} + \frac{\text{PE} \cdot \text{DE}}{\text{BE}} \times \text{h. l. } \frac{\text{BE}^2 + \text{BE} \cdot \text{BP} - \text{PE} \cdot \text{DE}}{\text{PE} \cdot \text{BE} - \text{PE} \cdot \text{DE}} \\ = \frac{sc}{ee} \cdot a - x \times d - g + \frac{cg}{e} \times \text{h. l. } \frac{ec + de - cg}{eg - cg}. \end{aligned}$$

These quantities being collected together with their proper signs, and contracted, we have

$$s \times \left\{ \begin{array}{l} \frac{ab}{d} + c \cdot \frac{ad - qq - dx}{ee} + x \times \text{hyp. log. } \frac{ag + qq - bx}{b + d \cdot x} + \\ \frac{cvg \cdot a - x}{e^3} \times \text{h. l. } \frac{ec + de - cg \cdot acg + eqq + a^2 x - bcx}{gg \cdot ee - cc \cdot a - x} \end{array} \right\}$$

for the whole attraction in the direction PE.

9. Having now obtained a general formula for the measure of the attraction in any sort of triangle, if the particular values of the letters be substituted which any practical case may require, and the fluxion of this attraction be put = 0, the root of the resulting equation will be the required height from the bottom of the hill.

10. But for a more particular solution in simpler terms, let us suppose the triangle ABC to be isosceles, in which case we shall have $d=e$, and $g=2b=2c$, and then the above general formula will become

$$s \times \left\{ \begin{array}{l} \frac{2ad - qq - dx}{dd} b + x \times \text{h. l. } \frac{2ab + qq - bx}{b + d \cdot x} \\ + \frac{a-x}{d^3} \cdot 2b^3 \times \text{h. l. } \frac{2ab^2 + dq^2 - b^2 - a^2 \cdot x}{2b^2 \cdot a - x} \end{array} \right\}$$

for the value of the attraction in the case of the isosceles triangle, where q^2 is = $\sqrt{4a^2b^2 - 4ab^2x + d^2x^2}$. And the fluxion of this expression being equated to 0, the equation will give the relation between a and x for any values of b and d , by a process not very troublesome.

11. Now it is probable that the relation between a and x , when the attraction is greatest, will vary with the various relations between b and d , or between b and a . Let us therefore find the limits of that relation, between which it may always be taken, by using two particular extreme cases, the one in which the hill is

very steep, and the other in which it is very flat, or a very small in respect of b or d .

12. And first let us suppose the triangular section to be equilateral; in which case the angle of elevation is 60° , which being a degree of steepness that can scarcely ever happen, this may be accounted the first extreme case. Here then we shall have $d = 2b = \frac{2}{3}a\sqrt{3}$, and the formula in Art. 10, will become $s \times : \frac{2a - r - x}{2} + x \times$ h. l. $\frac{2a + 2r - x}{3x} + \frac{a - x}{4} \times$ h. l. $\frac{a + 2r + x}{a - x}$ for the value of the attraction in the case of the equilateral triangle, in which r is $= \sqrt{a^2 - ax + x^2}$.

13. Or if we take $x = na$, where n expresses what part of a is denoted by x , the last formula will become $sa \times : 1 - \frac{1}{2}n - \frac{1}{2}\sqrt{1 - n + n^2} + n \times$ h. l. $\frac{2 - n + 2\sqrt{1 - n + n^2}}{3n} + \frac{1 - n}{4} \times$ h. l. $\frac{1 + n + 2\sqrt{1 - n + n^2}}{1 - n}$ for the case of the equilateral triangle.

14. To find the maximum of the expression in the last article, put its fluxion = 0, and there will result this equation $1 + \frac{1 + n}{\sqrt{1 - n + n^2}} = 2$ h. l. $\frac{2 - n + 2\sqrt{1 - n + n^2}}{3n} - \frac{1}{2}$ h. l. $\frac{1 + n + 2\sqrt{1 - n + n^2}}{1 - n}$; the root of which is $n = .251999$. Which shews that, in the equilateral triangle, the height from the bottom to the point of greatest attraction, is only

only $\frac{1}{500}$ th part more than $\frac{1}{4}$ of the whole altitude of the triangle. And this is the limit for the steepest kind of hills.

15. Let us find now the particular values of the measure of attraction arising by taking certain values of n varying by some small difference, in order to discover what part of the greatest attraction is wanting by observing at different altitudes.

16. And first using the value of n ($\cdot 251999$) as found in the 14th article, the general formula in Art. 13, gives $sa \times 1\cdot 0763700$ for the measure of the greatest attraction.

17. If $n = \frac{3}{10}$, or $x = \frac{3}{10}a$; the same formula gives $\frac{sa}{20} \times : 17 - \sqrt{79} + 6 \text{ h. l. } \frac{17 + 2\sqrt{79}}{9} + \frac{7}{2} \text{ h. l. } \frac{13 + 2\sqrt{79}}{7} = sa \times 1\cdot 0702512$ for the attraction at $\frac{3}{10}$ of the altitude, which is something less than the other.

18. If $n = \frac{4}{10} = \frac{2}{5}$; the formula gives $\frac{sa}{20} \times : 16 - \sqrt{76} + 8 \text{ h. l. } \frac{8 + \sqrt{76}}{6} + 3 \text{ h. l. } \frac{7 + \sqrt{76}}{3} = sa \times 1\cdot 0224232$ for the attraction at $\frac{4}{10}$ or $\frac{2}{5}$ of the altitude; less again than the last was.

19. If $n = \frac{5}{10} = \frac{1}{2}$; the formula gives $\frac{1}{4}sa \times : 3 - \sqrt{3} - 2 \text{ h. l. } 3 + \frac{5}{2} \text{ h. l. } \frac{3 + 2\sqrt{3}}{3} = sa \times \cdot 9340963$ for the attraction at half way up the hill; still less again than the last.

20. If $n = \frac{6}{10} = \frac{3}{5}$; the formula gives $\frac{sa}{20} \times : 14 - \sqrt{76} + 12 \text{ h. l. } \frac{7 + \sqrt{76}}{9} + 2 \text{ h. l. } \frac{8 + \sqrt{76}}{2} = sa \times \cdot 8109843$ for the attraction at $\frac{6}{10}$ or $\frac{3}{5}$ of the altitude from the bottom; being still less than the last was. And thus the quantity of attraction is continually less and less the higher we ascend up the hill above the $\cdot 251999$ part, or in round numbers $\cdot 252$ part of the altitude. Let us now descend, by trying the numbers below $\cdot 252$; and first,

21. If $n = \cdot 25 = \frac{1}{4}$; the same formula in Art. 13, gives $\frac{1}{8} sa \times : 7 - \sqrt{13} + 2 \text{ h. l. } \frac{7 + 2\sqrt{13}}{3} + \frac{3}{2} \text{ h. l. } \frac{5 + 2\sqrt{13}}{3} = sa \times 1\cdot 0763589$ for the attraction at $\frac{1}{4}$ of the altitude; and is very little less than the maximum.

22. If $n = \frac{2}{10} = \frac{1}{5}$; the formula gives $\frac{1}{10} sa \times : 9 - \sqrt{21} + 2 \text{ h. l. } \frac{9 + 2\sqrt{21}}{3} + 2 \text{ h. l. } \frac{3 + \sqrt{21}}{2} = \frac{1}{10} sa \times : 9 - \sqrt{21} + 2 \text{ h. l. } \frac{23 + 5\sqrt{21}}{2} = sa \times 1\cdot 0684622$ for the attraction at $\frac{2}{10}$ or $\frac{1}{5}$ of the altitude; and is something less than at $\frac{1}{4}$ of the altitude.

23. If $n = \frac{1}{10}$; the formula gives $\frac{sa}{20} \times : 19 - \sqrt{91} + 2 \text{ h. l. } \frac{19 + 2\sqrt{91}}{3} + \frac{2}{9} \text{ h. l. } \frac{11 + 2\sqrt{91}}{9} = sa \times \cdot 9986188$ for the attraction at $\frac{1}{10}$ of the altitude; still less than the last was. And, lastly,

24. If $n = 0$, or the point be at the bottom of the
hill;

hill; the formula gives $\frac{1}{4}sa \times 2 + \text{h. l. } 3 = sa \times .7746531$ for the attraction at the bottom of the hill; which is between $\frac{2}{3}$ and $\frac{3}{4}$ of the greatest attraction, being something greater than $\frac{2}{3}$ but less than $\frac{3}{4}$ of it.

25. The annexed table exhibits a summary of the

$\frac{6}{100}$	8109843	$\frac{1}{4}$
$\frac{5}{100}$	9340963	$\frac{2}{13}$
$\frac{4}{100}$	10224232	$\frac{1}{20}$
$\frac{3}{100}$	10702512	$\frac{1}{100}$
$\frac{2 \frac{1}{2}}{1000}$	10763700	0
$\frac{1}{4}$	10763589	$\frac{1}{97852}$
$\frac{2}{100}$	10684622	$\frac{1}{134}$
$\frac{1}{100}$	9986188	$\frac{1}{14}$
0	7746531	$\frac{2}{7}$

calculations made in the preceding articles; where the first column shews at what part of the altitude of the hill the observation is made; the second column contains the corresponding numbers which are proportional to the attraction; and the third column shews what part

of the greatest attraction is lost at each respective place of observation, or how much each is less than the greatest.

26. Having now so fully illustrated the case of the first extreme, or limit, let us search what is the limit for the other extreme, that is, when the hill is very low or flat. In this case b is nearly equal to d , and they are both very great in respect of a ; consequently the formula for the attraction in Art. 10, will become barely $s \times x \times \text{h. l. } \frac{2a-x}{x} + 2 \cdot \overline{a-x} \times \text{h. l. } \frac{2a-x}{a-x}$; the fluxion of which being put = 0, we obtain $0 = \text{h. l. } \frac{2a-x}{x} - 2 \text{h. l. } \frac{2a-x}{a-x} = \text{h. l. } \frac{2a-x}{x} - \text{h. l. } \frac{2a-x}{a-x} \Big|^2 = \text{h. l. } \frac{\overline{a-x}^2}{x \cdot 2a-x}$;

hence therefore $\overline{a-x}^2 = x \cdot 2a-x$, and $x = a \times 1 - \sqrt{\frac{1}{2}} =$

·2929a. Which shews that the other limit is $\frac{29}{100}$; that is, when the hill is extremely low, the point of greatest attraction is at $\frac{29}{100}$ of the altitude, like as it is at $\frac{25}{100}$ when the hill is very steep. And between these limits it is always found, it being nearer to the one or the other of them, as the hill is flatter or steeper.

27. Thus then we find that at $\frac{1}{4}$ of the altitude, or very little more, is the best place for observation, to have the greatest attraction from a hill in the form of a triangular prism of an indefinite length. But when its length is limited, the point of greatest attraction will descend a little lower; and the shorter the hill is, the lower will that point descend. For the same reason, all pyramidal hills have their place of greatest attraction a little below that above determined. But if the hill have a considerable space flat at the top, after the manner of a frustum, then the said point will be a little higher than as above found. Commonly, however, $\frac{1}{4}$ of the altitude may be used for the best place of observation, as the point of greatest attraction will seldom differ sensibly from that place. And when uncommon circumstances may produce a difference too great to be intirely neglected, the observer must exercise his judgment in guessing at the necessary change he ought to make in the place of observation, so as to obtain the best effect which the concomitant circumstances will admit of.

